INVARIANCE AND PARALLEL SUMS

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Abstract. In this paper the notions of invariance and parallel sums as defined by Anderson and Duffin for matrices (Journal of Mathematics Analysis and Applications **26**, 576-594 (1969)) are generalized to von Neumann regular rings.

Introduction

Let R be a von Neumann regular algebra over any commutative ring. For $a, b \in R$, we set $a^- = \{x \in R \mid axa = a\}$, and define the parallel sum $P(a, b)$ of a, b as $P(a, b) = a(a + b)^- b$. This notion, introduced by Anderson and Duffin using the Moore Penrose inverses, arose from the notion of the impedance matrix of two n-port electrical networks connected in parallel ([2]). Anderson and Duffin obtained many interesting properties of the parallel sum of a pair of Hermitian semidefinite matrices.

The concept of parallel summability was extended by Rao and Mitra who proved similar results of Anderson and Duffin in a general setting replacing the Moore-Penrose inverse by a generalized inverse, also known as inner inverse ([7]).

This work of Rao and Mitra leads naturally to the question of determining $a, b \in R$ such that $a^- + b^-$ is precisely the class c^- for a single element c. Odell and Mitra ([6]) showed that for matrices a, b, c over any field, $c^- = a^- + b^-$ if and only if $\{c\} = c(a + b)^- b$. In an earlier paper ([1]), we proved that in any von Neumann regular ring R, if such an element c exists then it must be unique. Recently T.H Lee proved that if $a^= \subseteq b^=$ then $a = b$, where $a^= = \{x \in R \mid axa = a \text{ and } xax = x\}$.

In this paper we show, for elements a, b, c in a von Neumann regular ring R, that $c^- = a^- + b^-$ if and only if $a(a + b)^- b$ is invariant with value equal to c, when

(1) a and b are commuting idempotent elements,
(2) a is an idempotent and b a unit, or
(3) the ring R is an abelian regular ring.

Throughout the paper we will denote the set of invertible elements of a ring R by $U(R)$. We will write $c = a(a + b)^- b$ instead of $\{c\} = a(a + b)^- b$.

1. Preliminaries

Throughout R denotes any von Neumann regular algebra with identity over a commutative ring. For $a \in R$, let a^- and Ian(a) denote the set of inner inverses and inner annihilators of a, respectively. More explicitly, we have $a^- = \{x \in R \mid axa = a\}$ and $\text{Ian}(a) = \{x \in R \mid axa = 0\}$. Let us first give a complete description of the class a^- (c.f. [3] p.40, Corollary 1).

Lemma 1. *For an element a in a regular ring R and $a_0 \in a^-$, $a^- = \{a_0 + t - a_0ataa_0 \mid t \in R\} = a_0 + \mathrm{Ian}(a)$.*

If an expression involving inner inverses of certain elements of R remains unaltered by plugging different values of inner inverses, we call that expression invariant. We can now state our first proposition.

Proposition 2. *Let $a, b, c \in R$ and suppose R is prime. Then ba^-c is invariant if and only if $b \in Ra$ and $c \in aR$.*

Proof. Let $a_0 \in a^-$ and suppose that ba^-c is invariant. We then have $ba^-c = ba_0c = b\{a_0 + t - a_0ataa_0 \mid t \in R\}c$. Therefore, for all $t \in R$ we have $btc = ba_0ataa_oc$. Since aa_0 and a_0a are both idempotents, replacing t by $(1 - a_0a)t$ and $t(1 - aa_0)$ successively, we get $b(1 - a_0a)tc = 0$ and $bt(1 - aa_0)c = 0$. The fact that the ring R is prime leads to the desired conclusion.

Conversely, if $b \in Ra$ and $c \in aR$, say $b = \beta a$ and $c = a\gamma$ we get $ba^-c = \beta aa^-a\gamma$, and hence ba^-c is invariant. □

In general, we have the following proposition.

Proposition 3. *Let $a, b \in R$. Then ba^-b is invariant if and only if $b \in Ra \cap aR$.*

Proof. This is straightforeward, since R is a von Neumann regular ring and hence semiprime. □

For a in R, let $a^= = \{x \in R \mid a = axa \text{ and } xax = x\}$. The elements of $a^=$ are called reflexive inverses of a. The followig lemma is also stated without proofs. The reader can easily verify the statements.

Lemma 4. *Let $a, b, c \in R$ be such that $c^- = a^- + b^-$ and let a_0, b_0 be elements in $a^=, b^=$, respectively. Then we have*

1. *ca^-c and cb^-c are invariant.*
2. *For all $t \in R$ we have that $ctc = ca_0ataa_0c = cb_0btbb_0c$.*
3. *For all $t \in R$ we have that $ctc = ca_0atc = ctaa_0c = cb_0btc = ctbb_0c$.*
4. *$ca^-c = ca^=c$ and $cb^-c = cb^=c$.*
5. *If we assume that R is prime we get $c = ca_0a = aa_0c = cb_0b = bb_0c$.*
6. *If we assume that R is a prime ring we have, for all $t \in R$ $ctc = ca_0atc = ctaa_0c = cb_0btc = ctbb_0c$.*
7. *If we assume that $a = 1$ and $bc = cb$ we get, without any assumption on R, that $cb = c^2(b + 1)$.*

2. Parallel sums

We prove an analogue of Odell and Mitra's result which states that for any matrix ring R over a field if $a, b, c \in R$ satisfying $c^- = a^- + b^-$, then the expression $a(a + b)^-b$ is invariant and the common value is c.

Lemma 5. *Let $a, b, c \in R$. If $c^- = a^- + b^-$ then ca^-c and cb^-c are invariant and hence $c \in Ra \cap Rb \cap aR \cap bR$.*

Proof. The proof follows by invoking Proposition 3. □

Since R is a regular ring, we know that there exist idempotents $g, h \in R$ such that $aR \cap bR = gR$ and $Ra \cap Rb = Rh$. Note that the preceding lemma shows that, when $c^- = a^- + b^-$, we have $cR \subseteq gR$ and $Rc \subseteq Rh$.

Lemma 6. *Let $a, b, c \in R$ and let $c^- = a^- + b^-$. Let g, h be idempotents in R such that $aR \cap bR = gR$ and $Ra \cap Rb = Rh$. Then, $cxc = 0$ for all $x \in R$, if and only if $hxg = 0$ for all $x \in R$.*

Proof. Let $c_0 \in c^- = a^- + b^-$ then there exist $a_0 \in a^-$ and $b_0 \in b^-$ such that $c_0 = a_0 + b_0$. Then Lemma 1 implies that $\mathrm{Ian}(c) = \mathrm{Ian}(a) + \mathrm{Ian}(b)$. So if $x \in \mathrm{Ian}(c)$, then there exist $y \in \mathrm{Ian}(a)I$ and $z \in \mathrm{Ian}(b)$ such that $x = y + z$. Since $g \in aR \cap bR$ and $h \in Ra \cap Rb$ we get $hxg = hyg + hzg = 0$, as required.

Now sppose that $hxg = 0$. Since $cR \subseteq aR \cap bR = gR$ and $Rc \subseteq Ra \cap Rb = Rh$ we have $cxc \in RhxgR = 0$. □

Under the notations as in the above lemma, we have seen that $cR \subseteq gR$ and $Rc \subseteq Rh$. Assuming that the ring is prime and twosided self-injective we can also prove the opposite inclusions. Somewhat more generally, if a ring R is such that for any element $a \in R$ we have $l(ra)) = Ra$ and $r(l(a)) = aR$ (for instance if R is right and left self-injective) we say that R satisfies the double annihilator conditions for elements. Assuming that R satisfies this condition, we prove the following proposition.

Proposition 7. *Let $a, b, c \in R$, and $c \neq 0$. Let R be prime satisfying the double annihilator conditions. If $c^- = a^- + b^-$ then $Ra \cap Rb = Rc$ and $aR \cap bR = cR$.*

Proof. Let $g = g^2 \in R$ and $h = h^2 \in R$ be such $aR \cap bR = gR$ and $Ra \cap Rb = Rh$. We then have by Lemma 5 that $cR \subseteq gR$ and $Rc \subseteq Rh$. Let us first remark that $g \neq 0$. Making use of the previous lemma we get $hr(c)g \subseteq h\mathrm{Ian}(c)g = 0$ hence $r(c)g \subseteq r(h)$ and since R has the double annihilator condition this leads to $Rh = l(r(h)) \subseteq l(r(c)g)$. Notice that since R is prime and $g \neq 0$, we have that $xr(c)g = 0$ implies that $xr(c) = 0$ i.e. $l(r(c)g) \subseteq l(r(c))$. We thus get $Rh \subseteq l(r(c)) = Rc$ i.e. $Ra \cap Rb = Rc$.

We get, similarly, that $aR \cap bR = gR \subseteq cR$. □

3. The equation $c^- = a^- + b^-$

It was proved by Mitra and Odell that for matrices a, b, c over a field, $c^- = a^- + b^-$ if and only if $\bar{=}a(a+b)^-b$ holds. Our purpose is to analyze this relation between the above equations for certain special elements in regular rings. It is easy to show that this equivalence between two equations always holds for commuting invertible elements a and b.

We start with the case when a and b are idempotents. Recall that $U(R)$ denotes the set of invertible elements of R.

Lemma 8. *Let $e = e^2$, $f = f^2$ be two commuting idempotents of a ring R with $2 \in U(R)$. Then $2(ef)^- = e^- + f^-$.*

Proof. Assume that $x \in e^-, y \in f^-$. Then $(\frac{ef}{2})(x+y)(\frac{ef}{2}) = (\frac{f}{2})exe(\frac{f}{2}) + (\frac{e}{2})fyf(\frac{e}{2}) = (\frac{ef}{4}) + (\frac{ef}{4}) = (\frac{ef}{2})$. This shows that $e^- + f^- \subseteq (\frac{ef}{2})^-$. Let us now show the converse inclusion. Since $ef = fe$, ef is an idempotent and we have $2(ef)^- = 2\{1 - t + eftef \mid t \in R\}$ and also $e^- = \{1 - u + eue \mid u \in R\}$

$f^- = \{1 - v + fvf \mid v \in R\}$. For t fixed in R, choose $u = 2t$ and $v = 2ete$ in these descriptions. This leads immediately to the required inclusion. $\square$

We give below some consequences of the equation $c^- = e^- + f^-$, where e, f, c are elements in a prime ring with e, f idempotents.

Lemma 9. *Let c be a regular element and e, f be idempotent elements in a prime ring R (R is not necessarily regular). If $c^- = e^- + f^-$, and $ef = fe$, then*

$$c = e(e + f)^- f.$$

Proof. First, we remark that since $c \in eR \cap fR$ and $c \in Re \cap Rf$ we have that $ec = c = ce = fc = cf$. Multiplying the equation $c^- = e^- + f^-$ by c on both sides we get $c = 2c^2$. Multiplying the equation $c^- = e^- + f^-$ on both sides by ef, we have $efc^-ef = 2ef$. Since $1 \in e^- \cap f^-$, we get $2 \in c^-$ and hence $c^- = \{2 - t + 4ctc \mid t \in R\}$. Plugging this value in the previous equality we get $eftef = 4efctcef = 4ctc$. Taking $t = e$ we are lead to

$$ef = 4cec = 4c^2 = 2(2c^2) = 2c \qquad (1)$$

Let $x \in (e+f)^-$ we must show that $c = exf$. Multiplying the equality $(e+f)x(e+f) = e+f$ on both sides by ef we obtain $ef(e+f)x(e+f)ef = ef(e+f)ef$. This gives $4efxef = 2ef = 4c$. We consider 2 cases.

First if char $(R) \neq 2$ the previous equality leads to $efxef = c$. Starting back from $(e+f)x(e+f) = e+f$ and multiplying this equation on the left by ef and on the right by e we get $2efx(e+ef) = 2ef$. This leads to $efxe+efxef = ef$ i.e. $efxe+c = 2c$ and hence $efxe = c$. Similarly, multiplying $(e + f)x(e + f) = e + f$ by ef on the right and e on the left we are lead to $exef = c$. By symmetry of our equation in e and f we also get $efxf = c = fxef$. Now multiplying again $(e+f)x(e+f) = e+f$ by e on the left and f on the right we have $(e + ef)x(ef + f) = 2ef = 4c$. This gives $exef + efxef + exf + efxf = 4c$. Taking into account all the values of c we have $exf = c$, as desired.

Now if char$(R) = 2$, equation (1) above leads $ef = 0$ and also $c = cef = 0$. If $x \in (e + f)^-$ then $(e + f)x(e + f) = e + f$. Multiplying this equality on the left by e and on the right by f we get $exf = 0 = c$, as required. $\square$

We are now ready to state one of the main theorems.

Theorem 10. *Let a, b, c be regular elements of a prime ring R. Suppose that $ab = ba$ and that one of the following conditions holds:*

(a) *$a, b \in U(R)$*
(b) *$a = u$ and $b = e$ where $u \in U(R)$ and $e = e^2$*
(c) *If $2 \in U(R)$ and $a = e, b = f$ are commuting idempotents.*

Then, $c^- = a^- + b^-$ if and only if $c = a(a + b)^- b$.

Proof. (a) This is left to the reader.

(b) Since $(ub)^- = b^- u^{-1}$, we can assume, without any lost of generality, that $a = 1$ and $b = ue$, where $ue = eu$. Let us then suppose that $c^- = 1 + (ue)^- = 1 + e^- u^{-1}$. This shows that $ec^-e = \{e + eu^{-1}\}$ is constant. Using the description of c^- defined in Lemma 1 we easily obtain that, for any $t \in R$ we have $ete = (1 + u^{-1})ctc(1 + u^{-1})$. Since $1 + u^{-1} \in C^-$, both $c(1 + u^{-1}) = c(e + u^{-1})$ and $(1 + u^{-1})c = (e + u^{-1})c$ are idempotents, and taking $t = 1 - (e + u^{-1})c$ we obtain $e = (1 + u^{-1})c = c(e + u^{-1})$. We solve this equation for c. Let us recall the well known fact that, for regular elements $a, b \in R$, satisfying $ba^-a = \{b\}$, any solution of $xa = b$ is such that $\{x\} = ba^-$ (cf. [3] Chapter 2 Theorem 2). In our case the consistency condition, namely $e(e + u^{-1})^-(e + u^{-1}) = e$ is satisfied and hence $c = (e + u^{-1})^-e = (eu + 1)^-ue = ue(1 + ue)^-$.

Let us suppose that $c = ue(1 + ue)^- = (1 + ue)^-ue$. If $x_0 \in (1 + ue)^-$, then $c = uex_0$ and Lemma 1 leads to the fact that for any $t \in R$, we have $uet = c(1+ue)t(1+ue)x_0$. Multiplying on the right by $1 + eu$ leads to $uet(1 + eu) = c(1 + ue)t(1 + eu)$ and hence, by primeness of R, we conclude that $ue = c(1 + eu) = c + cu$. Since $c = ce = ce^-e$ we get $ue = cu + ce^-e$ and $ceu + ce^-e = ue$ i.e. $c(1 + e^-u^{-1})ue = ue$. Cancelling u and right multiplying by c we get $c(1 + e^-u^{-1})c = c$ which shows that $1 + e^-u^{-1} \in c^-$. So we indeed have $1 + (eu)^- \subseteq c^-$. We now prove the converse inclusion. We have seen above that $c(1 + e^-u^{-1})e = e$. By symmetry we also have $(1 + e^-u^{-1})c = e$, and multiplying the equality $cc^-c = c$ by $(1 + e^-u^{-1})$ on both sides, we get $ec^-e = (1 + e^-u^{-1})e = e(1 + e^-u^{-1})e$. Now, using this equality, we compute $eu(c^- - 1)eu = uec^-eu - u^2e = ue(1 + e^-u^{-1})eu - u^2e = ue$. This shows that $c^- - 1 \subseteq (ue)^-$. This completes the proof of the reverse inclusion.

(c) If $c^- = e^- + f^-$ then Lemma 9 implies that $c = e(e + f)^-f$. To prove the converse implication we multiply the equality $(e + f)(e + f)^-(e + f) = e + f$ by e on the left and by f on the right and, since $ec = c = cf$, we obtain $4c = 2ef$and hence $2(ef)^- = c^-$. Lemma 8 gives the desired conclusion. □

We will now consider the case of a regular ring in which the idempotents are all central. Such rings are called abelian regular rings. A regular ring R is abelian if and only if it is strongly regular i.e. for any $x \in R$ there exists $y \in R$ such that $x^2y = x$. These rings are automatically unit regular and their elements can be presented as a product of an idempotent by a unit (cf. [G], for more details).

First, let us use our regular abelian hypothesis to reformulate our problem in terms of idempotents and invertible elements. We state without proof the following obvious lemma

Lemma 11. *Let R be an abelian regular ring and let $a = eu, b = fv, c = wg$, where e, f, g are idempotents and u, v, w are invertible elements. Then $c^- = a^- + b^-$ if and only if $g^- = (w^{-1}ue)^- + (w^{-1}vf)^-$*

So in an abelian regular ring, while considering $c^- = a^- + b^-$ we will assume that c is an idempotent, $a = ue$, $b = vf$ where e, f are idempotents, and u, v are invertible elements.

In the following results we collect basic properties connected to our problem.

Lemma 12. *Let e, f, c be idempotents in an abelian regular ring and $u, v \in U(R)$. Then the following are equivalent*

(i) $c^- \subseteq u^{-1}e^- + v^{-1}f^-$.

(ii) $\mathrm{Ian}(c) \subseteq \mathrm{Ian}(e) + \mathrm{Ian}(f)$ *and* $1 \in u^{-1}e^- + v^{-1}f^-$.

(iii) $(1 - c)R \subseteq (1 - e)R + (1 - f)R$ *and* $1 \in u^{-1}e^- + v^{-1}f^-$.

(iv) $\exists x, y \in R$ *such that* $1 - c = (1 - e)x + (1 - f)y$ *and* $c^- \cap (u^{-1}e^- + v^{-1}f^-) \neq \emptyset$

Under any one of the above statements, we have $ef = ef(u^{-1} + v^{-1})$ and $c = c(u^{-1} + v^{-1})$.

Proof. (i)⇔(ii)⇔(iii): It is enough to use Lemma 1, and the remark that $\mathrm{Ian}(ue) = \mathrm{Ian}(e) = (1 - e)R$ with $1 \in c^-$.

(iii)⇒(iv): This is obvious.

(iv)⇒(i) The first hypothesis in (iv) implies that $(1-c)R \subseteq (1-e)R + (1-f)R$ which exactly means that $\mathrm{Ian}(c) \subseteq \mathrm{Ian}(e) + \mathrm{Ian}(f)$. By our hypothesis there exists an element $z = z_1 + z_2$ with $z \in c^-$, $z_1 \in u^{-1}e^-$, and $z_2 \in v^{-1}f^-$. The required inclusion $c^- \subseteq u^{-1}e^- + v^{-1}f^-$ is then a direct consequence from the fact that $c^- = z + \mathrm{Ian}(c)$, $u^{-1}e^- = z_1 + \mathrm{Ian}(e)$, and $v^{-1}f^- = z_2 + \mathrm{Ian}(f)$.

To prove the additional statements notice that $ee^- = e$, $ff^- = f$, and $1 \in u^{-1}e^- + v^{-1}f^-$. We then easily get $ef = ef(u^{-1} + v^{-1})$ and multiplying this equality by c gives $c = c(u^{-1} + v^{-1})$. □

In the following theorem we will assume that $2 \in U(R)$.

Theorem 13. *Let e and f be idempotents in an abelian regular ring R and $u, v, 2 \in U(R)$. Then $c^- = (ue)^- + (vf)^-$ if and only if $c = ue(ue + fv)^- vf$ and $ef = ef(u^{-1} + v^{-1})$.*

Proof. Let us assume that $c^- = (ue)^- + (vf)^-$.

Lemma 12 implies that $= ef(u^{-1}+v^{-1}) = ef$. We thus have $efc^- = ef(u^{-1}e^- + v^{-1}f^-) = ef(u^{-1} + v^{-1}) = ef$, by Lemma 12. Now replacing c^- by $\{1 - t + ctc\}$, we conclude that, for any $t \in R$, we have $eft = ef(ctc) = ctc$. Taking $t = 1$, we get $c = ef$. First notice that by Lemma 12 we have $ef = ef(u^{-1} + v^{-1})$. Multiplying this equality respectively, by uv and vu, we get $efuv = ef(u + v)$ and $efvu = ef(v + u)$. This leads to

$$ef(uv) = ef(u + v) = ef(vu). \qquad (1)$$

Let x be any element in $(ue + fv)^-$. We want to show that $c = efuxv$. We have $(ue + vf)x(ue + vf) = ue + vf$. Multiplying by ef and using (1), we get $ef(u + v)x(u + v) = ef(u + v) = efuv = efvu$. This implies by using (1) again $efvuxvu = efvu$. This gives $efuxv = ef = c$, as desired.

Let us now show the converse implication. By hypothesis $c^2 = c = eu(eu + fv)^- fv = ef(ev^{-1} + fu^{-1})^-$. Since $ef(ev^{-1} + fu^{-1})^-$ is invariant for all choices of the inner inverses of $ev^{-1} + fu^{-1}$ we replace $(ev^{-1} + fu^{-1})^-$ by the reflexive inverses $(ev^{-1} + fu^{-1})^=$ and we obtain $c = ef(ev^{-1} + fu^{-1})^=$. We want to show that $c^- = u^{-1}e^- + v^{-1}f^-$. We first show the inclusion $u^{-1}e^- + v^{-1}f^- \subseteq c^-$. Notice first that $ce = c = cf$ and $ee^- = e$, $ff^- = f$. We then have $c(u^{-1}e^- + v^{-1}f^-)c = c(u^{-1}ef + v^{-1}ef)c = efc(u^{-1}+v^{-1})c = efc(fu^{-1}+ev^{-1})c = ef(ev^{-1} + fu^{-1})^=(ev^{-1} + fu^{-1})(ev^{-1} + fu^{-1})^= = ef(ev^{-1} + fu^{-1})^= = c$. This implies that $u^{-1}e^- + v^{-1}f^- \subseteq c^-$.

We now proceed to show that $c^- \subseteq u^{-1}e^- + v^{-1}f^-$. Since $u^{-1}e^- + v^{-1}f^- \subseteq c^-$, $u^{-1} + v^{-1} \in c^-$ and so $c^- \cap (v^{-1}f^- + u^{-1}e^-) \neq \emptyset$. Further more $u^{-1} + v^{-1} \in c^-$ implies $c = c(u^{-1} + v^{-1})$. By hypothesis $c = ef(ev^{-1} + fu^{-1})^-$. Multiplying this equality on both sides by $ev^{-1} + fu^{-1}$ and using the fact that $c = c^2 = ce = cf$ is central, we get $(u^{-1} + v^{-1})c(u^{-1} + v^{-1}) = c = ef(v^{-1} + u^{-1})$. Using our condition $ef = ef(u^{-1} + v^{-1})$, we obtain $c = ef$. On the other hand we have that $\frac{1}{2}((1-e)(1+f) + (1-f)(1+e)) = 1 - ef = 1 - c$. The implication $(iv) \Rightarrow (i)$ in Lemma 12 then shows that $c^- \subseteq u^{-1}e^- + v^{-1}f^-$, as desired. This concludes the proof. □

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References

[1] A. Alahmadi, S.K. Jain, A. Leroy, Regular elements determined by generalized inverses, Journal of Algebra and its Applications, J. Algebra Appl. **18** (7) (2019), 1950128

[2] W.N. Anderson and R.J. Duffin, Series and parallel addition of matrices, Journal of Mathematics analysis and applications **26**, 576-594 (1969).

[3] A. Ben-Israel, T. N.E. Greville, Generalized inverses Theory and applications, John Wiley and sons, New York (1974) (reprinted by Canadian Mathematical Society, 2003).

[4] K.R. Goodearl, von Neumann regular rings, second edition Krieger Publ. Co., Malabar, Florida, 1991.

[5] T.K. Lee, A note on inner and reflexive inverses in semiprime rings, to appear in J.A.A.

[6] S.K. Mitra, P.L. Odell, On parallel summability of matrices, Linear Algebra and its Applications (74)

[7] C.R. Rao, S.K. Mitra, Generalized Inverse of matrices and its Applications, Wiley, New York (1971).

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